Quantum Optics I
Lecture 5

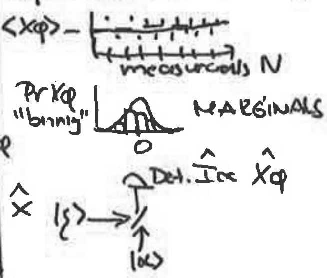
MEASUREMENT

Homodyne:
$X_\varphi = X_1 \cos(\varphi) + X_2 \sin(\varphi)$

$\langle X_\varphi \rangle$ – measurements N

Pr X_φ "binning" MARGINALS

Det. Ic $\hat{X}_\varphi$

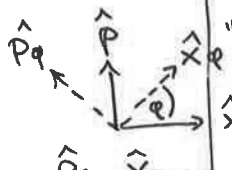


NOTE

Quantum State Tomography (continuous variable Q state tomography)

Wigner function: $W(\alpha, \alpha^*) = W(q,p)$

Quadrature distribution
$$\boxed{\begin{aligned}\hat{X}_\varphi &= \hat{x}\cos\varphi + \hat{p}\sin\varphi \\ \hat{P}_\varphi &= -\hat{x}\sin\varphi + p\cos(\varphi)\end{aligned}}$$



$\hat{P}_\varphi = \hat{X}_{\varphi+\pi/2}$

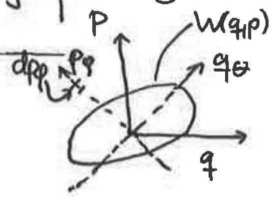
we need to compute the distribution (Prob-density function) for making a measurement with outcome X_φ

$$\boxed{*}\quad \underbrace{P_\Phi(X_\varphi)}_{\text{Probability distribution of } X_\varphi} = \int_{-\infty}^{\infty} W(\underbrace{X_\varphi\cos\varphi - p_\varphi\sin\varphi}_{\hat{x}^*},\ \underbrace{X_\varphi\sin\varphi + p_\Phi\cos\varphi}_{\hat{p}})\,dp_\varphi$$

X_φ: observable (outcome of measurement)

MARGINAL *

integration along "cut"



Note that $\langle \hat{X}_\varphi \rangle$ is expectation value is NOT sufficient to characterize the quantum state. We need the distribution.

$\boxed{*}$ key relationship between Wigner Function and homodyne signals

* Expression is called MARGINAL and gives prob. distribution

i.e $\left| \langle x_\varphi | \hat{X}_\varphi | x_\varphi \rangle \right|^2 \hat{=} P_\varphi(X_\varphi) \hat{=} \langle X_\varphi | \hat{\varsigma} | X_\varphi \rangle$ with X_φ Quadrature Eigenstate

expectation value with $|X_\varphi\rangle$

Note:

i.e $\hat{X}_\varphi |X_\varphi\rangle = X_\varphi |X_\varphi\rangle$ is an eigenstate.

* Original Wigner Function (Wigner 1932) *

$$W_\varsigma(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle q + \tfrac{1}{2}q' | \hat{\varsigma} | q - \tfrac{1}{2}q' \rangle e^{-ipq'}\, dq'$$

HISTORICAL DEFINITION 1932 WIGNER

$W^*(q,p) = W(q,p)$: real valued function.

$W(q,p)$ is a joint probability function for $\hat{p}, \hat{q}$ operator measurement outcomes. i.e

$$pr(q) = \underset{\text{eigenstate of position}}{\langle q|}\underset{\text{density matrix}}{\hat{\varsigma}}|q\rangle = \int_{-\infty}^{\infty} W(q,p)\,dp$$

likewise (ie $Pr(q, \theta = \pi/2 \hat{=} Pr(p)$):

$$pr(q,\theta=\pi/2) = pr(p) = \underset{\text{momentum Eigenstate}}{\langle p|}\hat{\varsigma}|p\rangle = \int_{-\infty}^{\infty} W(q,p)\,dq$$

MARGINALS.

WIGNER FUNCTIONS GIVES MOMENTUM & POSITION PROB. DISTRIBUTION (as before)

NOTE: the definition of MARGINALS defines WIGNER FCT *

For arbitrary rotation

$$pr(q,\theta) = pr(q_\theta) \hat{=} pr(X_\varphi) = \int_{-\infty}^{\infty} W(q\cos\theta - p\sin\theta,\ q\sin\theta + p\cos\theta)\,dp$$

$$\hat{=} \langle q | U(\theta) \hat{\varsigma} U^\dagger(\theta) | q \rangle$$

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Quantum Optics I

– Lecture 5.

Quantum State TOMOGRAPHY

(7.1) Refs: Vogel) Introduction to QO; Ulf LEONHARDT measuring the quantum state of light

Motivation: How to reconstruct a quantum state of light e.g. a squeezed state? $\varrho_{n,m} = \langle n|\varrho|m\rangle$

$\langle \hat{I}_{photodetector} \rangle \langle a^\dagger a \rangle = Tr[\varrho a^\dagger a] = \sum_n \varrho_{nn} \cdot n$

• Issue we cannot directly measure e.g. $\varrho_{nn} = \langle n|\varrho|n\rangle$ with a photodetector, i.e. "Photon resolving" measurements. Exception: (lately) cQED in dispersive regime allows measuring $p_n = \varrho_{nn}$ → Photon number statistics or QND in cQED → Haroche experiment $\varrho_{n,m}$

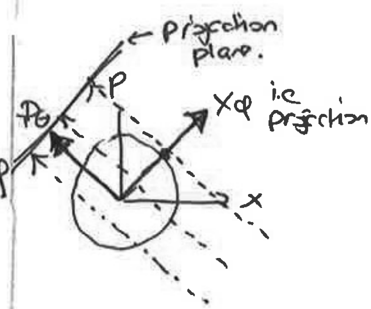
Key point: Measurement of $P_\Phi(x_\varphi)$ provides way to reconstruct $W(q,p)$ by noting that

$$\boxed{P_\Phi(x_\varphi) = \int_{-\infty}^{\infty} W(q,p)\,dp}$$

(marginal along axis $\hat{x}_\varphi$, $\hat{x}_{\varphi+\pi/2}$)

one can use inverse RADON transform to reconstruct the State (for rotationally symmetric states e.g Fock states it becomes simple Abel transform)

[Nobel '79 Medicine]



$$P(x_\varphi) = \int_{-\infty}^{\infty} W(q_\varphi \cos\varphi - p_\varphi \sin\varphi, \sin\varphi\, q_\varphi + \cos(\varphi) p)\, dp_\varphi$$

integral along a "cut" of Wigner function $W(q,p)$

→ Inverse transformation Radon 1917.

For rotationally symmetric state in p,q space:

$$\boxed{W(r) = -\frac{1}{\pi}\int_r^{\infty} \frac{dpr(x_\varphi)}{dx_\varphi}\,(x_\varphi^2 - r^2)^{-1/2}\, dx_\varphi \quad (x_\varphi = const.)}$$

Important comment: imperfect measurements will destroy reconstruction as vacuum state get's added as statistical mixture

Note: also the density matrix $\varrho_{n,m}$ can be reconstructed from the Wigner function.

$$\boxed{\varrho_{n,m} = \int \underbrace{W(q,p)}_{\text{Wigner function of state}} \cdot \overbrace{\underbrace{W_{nm}(q,p)}_{\text{representation of } \varrho = |n\rangle\langle m| \text{ projector}}}^{\text{Pattern Function}}\, dp\, dq}$$

Quantum Optics

Characteri(si)ng Quantum States

Covariance Matrix

Since squeezed states are gaussian states (with G-statistics)
4 numbers characterize their full statistics. $\{V_{XX}, V_{XY}, V_{YX}, V_{YY}\}$

$$V_{XY} = \frac{\langle \hat{X}\hat{Y} + \hat{Y}\hat{X}\rangle - 2\langle\hat{X}\rangle\langle\hat{Y}\rangle}{2\Delta^2 \hat{X}_{VAC}}$$

i.e. COVARIANCE MATRIX.

$$C(X_1, X_2)_{p,q} = \frac{1}{2} \langle X_p X_q + X_q X_p \rangle - \langle X_p \rangle \langle X_q \rangle \qquad p,q = 1,2$$

determines the entire state of the field (ie quantum state).

From covariance one can determine the amount of entanglement. ("Logarithmic negativity")

(Ch. 5.2 from Milburn Walls.)

Two Mode squeezed states : Wigner function

$$S(r) = e^{r\hat{a}^\dagger\hat{b}^\dagger - r\hat{a}\hat{b}} \qquad \text{as before}$$

Wigner function of state $S(r)|0_A 0_B\rangle$

Two mode characteristic function: (before one mode)

$$\boxed{\begin{aligned}\chi(y_1,y_2,t) &= \langle 0_A 0_B| e^{y_1\hat{a}^\dagger - y_2^*\hat{a}}\, e^{y_2\hat{b}^\dagger - y_2^*\hat{b}}|0_A 0_B\rangle \\ &= e^{-\frac{1}{2}|y_1|^2 - \frac{1}{2}|y_2|^2}\end{aligned}}$$

with
$$y_1 = \cosh(r)\,y - y_2^*\sinh(r)$$
$$y_2 = \cosh(r)\,y - y_2^*\sinh(r)$$
} squeezing operators for mode $\hat{a}, \hat{b}$

This yields now via Fourier transform of characteristic function

$$W(\alpha_1,\alpha_2,t) = \underbrace{\frac{1}{\pi^2}\int d^2y_1 \int d^2y_2\, \exp(y_1^*\alpha_1 - y_2\alpha_2^*)\,\chi(u_1,u_2,t)}_{\text{2D F.T.}}$$

$$= \frac{4}{\pi^2}\exp\left(-2|\alpha_1\cosh(r) - \alpha_2^*\sinh(r)|^2 - 2|\alpha_2\cosh(r) - \alpha_1^*\sinh(r)|^2\right)$$

Choose new variables:

$$\gamma_1 \equiv \alpha_1 - \alpha_2^*$$
$$\gamma_2 \equiv \alpha_1 - \alpha_2^*$$

$$\boxed{W(\gamma_1,\gamma_2) = \frac{1}{4\pi^2} e^{-\frac{1}{2}\left(\frac{|\gamma_1|^2}{e^{2r}} + \frac{|\gamma_2|^2}{e^{-2r}}\right)}}$$

Quantum Optics I

ref. Lvovsky RMP

Two Mode squeezed states (vacuum)

$$\boxed{S(\xi)_{\text{two-mode}} = e^{\left(r\hat{a}^{+}\hat{b}^{+} - r\hat{a}\hat{b}\right)}}$$ $\hat{a}, \hat{b}$: two modes.

$$|\psi\rangle = \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} (\tanh r)^n \underbrace{|n, n\rangle_{A\,B}}_{\text{Perfectly correlated photon numbers.}}$$

→ pair production.

wave-functions associated with the state:

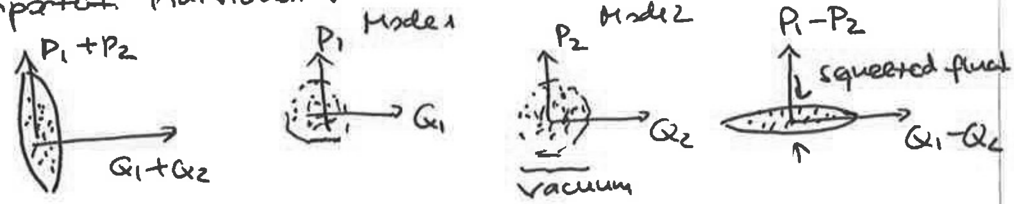
$$\langle Q_1, Q_2|\psi\rangle = \frac{1}{\sqrt{\pi}} e^{\left\{-\frac{1}{4}\underbrace{e^{2r}}_{\text{enhanced uncertainty}}(Q_1-Q_2)^2 - \frac{1}{4}\underbrace{e^{-2r}}_{\text{reduced fluctuations}}(Q_1+Q_2)^2\right\}}$$

$$\langle P_1, P_2|\psi\rangle = \frac{1}{\sqrt{\pi}} e^{\left\{-\frac{1}{4}\underbrace{e^{2r}}_{\text{enhanced}}(P_1+P_2)^2 - \frac{1}{4}\underbrace{e^{-2r}}_{\text{reduced fluctuations}}(P_1-P_2)^2\right\}}$$

⇒ $\hat{Q}_1+\hat{Q}_2$ and $\hat{P}_1-\hat{P}_2$ can simultaneously be measured with arbitrary precision.

As $r\to\infty$ the momenta and position are correlated and anti-correlated.

Important: individual variances are at the vacuum level.

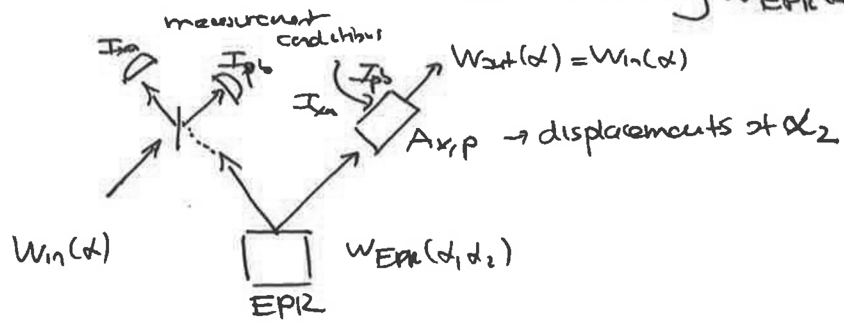


The states are due to the correlated nature called EPR-States (EPR paradox 1932)

Paradox: two systems are correlated. One system is measured and the correlation is such that value of variable is inferred with certainty e.g. $\left(|0_A 1_B\rangle \pm |1_A 0_B\rangle\right)\frac{1}{\sqrt{2}} = |\psi\rangle_{EPR}$

- Application: quantum teleportation and communication
  → see Kimble et al. & Braunstein.

- continuous quantum Teleportation takes state $W(\alpha)$ → teleports it to second location. using $W_{EPR}(\alpha)$



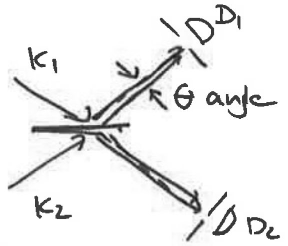
Quantum Optics I

Lecture 5

Nonclassical vs. non-classical predictions: interference of two photons. HONG-OU-MANDEL

Example of the differences between classical and non-classical fields.

Hong-Ou-Mandel interference



joint detection probability

$$p(x_a, x_b) \propto \langle E^-(x_a) E^-(x_b) E^+(x_b) E^+(x_a) \rangle$$

i.e. coincidence counts.

Intensities

Classically: $\cancel{E^- = \mathcal{E} e^{-i\omega t}}$ $\cancel{E^+ = \mathcal{E} e^{i\omega t}}$ $I_1 = \mathcal{E}^2 |\alpha_1|^2$, $I_2 = \mathcal{E}^2 |\alpha_2|^2$

$$p(x_a, x_b) \propto \left\{ \langle (I_1 + I_2)^2 \rangle - 2\langle I_1 I_2 \rangle \cos\left(2\pi \frac{(x_a - x_b)}{2}\right) \right\}$$

Note
$$\left| \begin{aligned} E^+(x_a) &= \frac{\mathcal{E}}{\sqrt{2}} (i \alpha_1 e^{ik_1 r} + \alpha_2 e^{ik_2 r}) \\ E^+(x_b) &= \frac{\mathcal{E}}{\sqrt{2}} (\underbrace{\alpha_1 e^{ik_1 r}}_{\text{input field 1}} + i \underbrace{\alpha_2 e^{ik_2 r}}_{\text{input field 2}}) \end{aligned} \right| \left. \right\} \begin{array}{l} \text{classical} \\ \hat{a}_1 \to \alpha_1 \\ \hat{a}_2 \to \alpha_2 \\ \text{c-Number} \end{array}$$

compute probability with these expressions.

visibility of the fringes

$p(x_a, x_b)$ max: $P_{max} = \langle (I_1 + I_2)^2 \rangle + 2 \langle I_1 I_2 \rangle$ (max.)

$p(x_a, x_b)$ min $P_{min} = \langle (I_1 + I_2)^2 \rangle - 2 \langle I_1 I_2 \rangle$ (min)

Visibility
$$\boxed{u = \frac{4\langle I_1 I_2 \rangle}{2(\langle I_1^2 \rangle + \langle I_2^2 \rangle + 2\langle I_1 I_2 \rangle)} < \frac{1}{2}}$$
since $\langle I_1^2 \rangle + \langle I_2^2 \rangle \geq 2\langle I_1 I_2 \rangle$

→ predicts low visibility $u = \left(\frac{P_{max} - P_{min}}{P_{max} + P_{min}} \right)$

Quantum mechanically Assume $|\psi\rangle = |1_A\rangle |1_B\rangle$ i.e. photon pairs that are indistinguishable

$$\left| \begin{aligned} E^+(x_a) &= \frac{\mathcal{E}}{\sqrt{2}} (i \hat{a}_1 e^{ik_1 r} + \hat{a}_2 e^{ik_2 r}) \\ E^-(x_b) &= \frac{\mathcal{E}}{\sqrt{2}} (\hat{a}_1 e^{ikr} + i \hat{a}_2 e^{ik_2 r}) \end{aligned} \right|$$

$$P(x_a, x_b) \propto \langle 1,1 | E^-(x_a) E^-(x_b) E^+(x_b) E^+(x_a) | 1,1 \rangle = \ldots$$

$$= \frac{1}{2} \mathcal{E}^4 \left\{ 1 - \cos\left[(\vec{k}_2 - \vec{k}_1) r_a - (\vec{k}_2 - \vec{k}_1) r_B \right] \right\} \Rightarrow \frac{P_{max} - P_{min}}{P_{max} + P_{min}} = 1 \; (!)$$

i.e. full contrast!

Quantum Optics I
- Lecture 5 -

Ref: Scully Q.O.
Milburn, G.O.

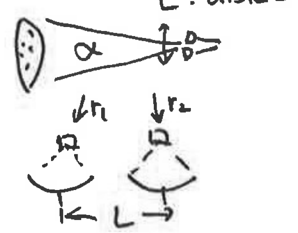
> " correlation between photons in two coherent beams of light" HBT 1956

→ homework

Experiment

Spatial coherence

L : distance



$g_2(r_1,r_2,T=0)$

$\alpha = \lambda / L_c$

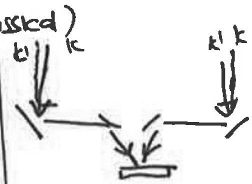
→ spatial coherence

Correlation (2nd)
$g^{(2)}(r_1,r_2,T)$

$$g^{(2)}(L,0) = \frac{\langle I(r_1,t), I(r_1+L,t+T)\rangle}{\langle I(r_1,t)\rangle \langle I(\underset{r_1+L}{r_2},t)\rangle}$$

Contrast to <u>Michelson Interferometer</u> (classical)

$$\boxed{g^{(1)} = \frac{\langle E^*(r_1,t) E(r_2,t+T)\rangle}{\langle |E(r_1,t)|^2\rangle^{1/2} \langle |E(r_2,t+T)|^2\rangle^{1/2}}}$$



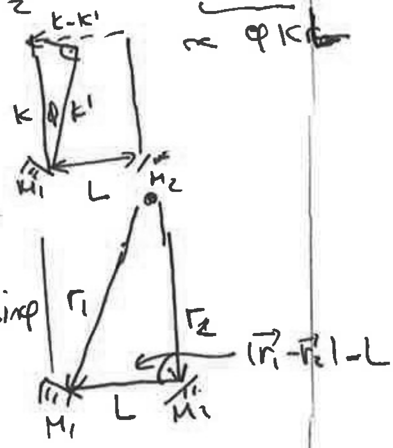
First order coherence

First order coherence.
→ sensitive to path length:
$\propto \cos(\underbrace{(k'+k)(r_1'-r_2)}_{\text{Path length difference}}) \cdot \cos(\frac{\pi r \varphi}{\lambda})$

Assume that $|E_k|^2 = |E_{k'}|^2 \approx I_0 \propto \langle (E_k^* + E_{k'}^*)(E_k + E_{k'})\rangle$

$$I \propto \langle E^* E\rangle = I_0 \{ 1 + \cos((k+k')\frac{(r_1-r_2)}{2}) \cos(\underbrace{\frac{(k-k')(r'-r^2)}{2}}_{\propto \varphi k})$$

→ one can determine distance to star/ galaxy. Problem: atmosphere in earth.



$$\boxed{(\vec{k}-\vec{k}')(\vec{r}_1-\vec{r}_2) = |\vec{k}-\vec{k}'| \underbrace{\cos\varphi}_{\approx 1} L = k \cdot \varphi \cdot L}$$

$|\vec{k}-\vec{k}'| \approx k \cdot \varphi = k \sin\varphi$

Note: interference present!

Return to 2nd order: (equal time).

$$\langle \overbrace{I(r_1,t) \cdot I(r_2,t)}^{\propto |E_k + E_{k'}|^2} \rangle = (|E_k|^2 + |E_{k'}|^2 + E_k E_{k'}^* e^{-i(k-k')r_2} + c.c.) \times (|E_k|^2 + |E_{k'}|^2 + E_k E_{k'}^* e^{-i(k-k')r} + cc)$$

→ interference.

$$= (|E_{k'}|^2 + |E_k|^2)^2 + |E_k|^2 |E_{k'}|^2 (e^{-i(\vec{k}-\vec{k}')(\vec{r}_1-\vec{r}_2)} + cc)$$

→ no term such as $\cos((k+k')\frac{(r_1-r_2)}{2})$ → fast oscillations (λ-scale)

But: $e^{-i(k-k')(\vec{r}_1-\vec{r}_2)} \propto 1$ ~~...~~ $\varphi\left(\frac{2\pi}{\lambda}\right) \cdot L$

Quantum Optics I

– Lecture 8 –

$$g^{(2)}(r_1, r_2, T) = \frac{\langle I(r_1,t) I(r_2, t+T)\rangle}{\langle I(r_1,t)\rangle \langle I(r_2, t+T)\rangle} = 1 + \left|\underbrace{g^{(1)}(r_1, r_2, T)}_{\text{First order coherence}}\right|^2$$

For independent detection events $g^{(2)} = 1$

* Interpretation: combines WAVE and PARTICLE character.
  "1": particle picture ("shot noise") and $g^{(1)}$: wave nature: "beat of random waves"

$g^{(2)} > 1$ CALLED BUNCHING OF PHOTONS

$g^{(2)} < 1$ called ANTI-BUNCHING OF PHOTONS

→ Quantum statistics of light.

$$g^{(2)}(r_1, r_1, T) = \frac{\langle E^-(t) E^-(t+T) E^+(t+T) E^+(t)\rangle}{\langle E^-(t+T) E^+(t+T)\rangle \langle E^-(t) E^+(t)\rangle}$$

$$\hat{=} \langle :I(t) I(t+T):\rangle / \langle I(t)\rangle\langle I(t+T)\rangle$$

Process of photon detection:

Quantum Mechanical description of photon counting: $\hat{E}^- \propto \hat{a}_k^+$ ∧ $\hat{E}^+ \propto \hat{a}_k$

absorptive detectors: $W(t) \propto \sum_f |\langle \underbrace{f}_{\text{final field state}} | \hat{E}^+(r,t) | \underbrace{i}_{\text{initial state}} \rangle|^2$

$\propto \hat{a}_k$ ie $W(t) \propto \langle i | \hat{E}^- \hat{E}^+ | i\rangle$

key point: photon is destroyed in detection process (ie $\hat{H} = \hat{a}_k$)

$$\boxed{W(t) = \sum_f \langle i | E^- \underbrace{|f\rangle\langle f|}_{\mathbb{1}} E^+ | i\rangle} \propto \langle i | \hat{E}^- \hat{E}^+ | i\rangle \propto \hat{a}_k^+ \hat{a}_k$$

OR: Simplified treatment $\langle \hat{I}(t)\rangle \propto \langle \hat{a}_k^+ \hat{a}_k\rangle = \langle \hat{n}_k\rangle$

Calculation of the correlations using e.g the $P(\alpha, \alpha^*)$ function

• COHERENT STATE: $|\alpha_0\rangle$

$$g^{(2)}(T) = \frac{\langle a^+(t) a^+(t+T) a(t+T) a(t)\rangle}{\langle a^+(t+T) a(t+T)\rangle \langle a^+(t) a(t)\rangle}$$

∧ $P(\alpha, \alpha^*) = \delta(\alpha - \alpha_0)$

$\hat{=} \langle a^+(t) a(t)\rangle^2$

Thus: $$g^{(2)}(T) = \frac{\int (\alpha^*)^2 \alpha^2 \, d^2\alpha \, \delta(\alpha - \alpha_0)}{\left(\int P(\alpha,\alpha^*) \delta(\alpha - \alpha_0) \alpha^* \alpha\right)^2} = \frac{|\alpha|^4}{(|\alpha|^2)^2} = 1$$

• THERMAL STATE $P(\alpha, \alpha^*) = \frac{1}{\pi \langle n\rangle} e^{-|\alpha|^2/\langle n\rangle}$

$$g^{(2)}(T) = \frac{\int P(\alpha, \alpha^*) |\alpha|^4}{\left(\int P(\alpha, \alpha^*) \alpha^* \alpha\right)^2} = 2$$: PHOTON BUNCHING.

* add to homework!

• SQUEEZED LIGHT $|\xi, \alpha\rangle$: squeezed coherent state

$g^{(2)}(T) < 1$ Nonclassical light. Photon are not independent any more.